New method of high-precision electron's anomalous magnetic moment calculation in QED

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AMM of the electron (theory and experiment)

The measured value [2011]:

a_e=0.00115965218073(28)

The most accurate prediction (T. Kinoshita et al. [2015]):

$$a_e = a_e(QED) + a_e(hadronic) + a_e(electroweak),$$

$$a_e(QED) = \sum_{n\geq 1}\left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n} = A_1^{(2n)} + A_2^{(2n)}(m_e/m_\mu) + A_2^{(2n)}(m_e/m_\tau) + A_3^{(2n)}(m_e/m_\mu, m_e/m_\tau)$$

a_e=0.00115965218164**3**(25)(23)(16)(763)

(α^{-1}=137.035999049(90) – from experiments with rubidium atoms)

Uncertainties come from:

$$A_1^{(8)},\ A_1^{(10)},\ a_e(hadronic) + a_e(electroweak),\ \alpha$$

T. Aoyama, M. Hayakawa, T. Kinoshita, M.Nio, Tenth-Order Electron Anomalous Magnetic Moment – Contribution of Diagrams without Closed Lepton Loops, Physical Review D, 2015, V. 91, 033006.

My method was developed for computing $A_1^{(2n)}$

Universal QED contributions

$$a_e = a_e(QED) + a_e(hadronic) + a_e(electroweak),$$

$$a_e(QED) = \sum_{n\geq 1}\left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n} = A_1^{(2n)} + A_2^{(2n)}(m_e/m_\mu) + A_2^{(2n)}(m_e/m_\tau) + A_3^{(2n)}(m_e/m_\mu, m_e/m_\tau)$$

- J. Schwinger [1948], analytically: $A_1^{(2)} = 0.5$
- R. Karplus, N. Kroll [1949] – with a mistake

  A. Petermann [1957], C. Sommerfield [1958], analytically:

$$A_1^{(4)} = -0.328478966\ldots$$

- ~1970…~1975, 3 loops, numerically:
  1. M. Levine, J. Wright.
  2. R. Carroll, Y. Yao.
  3. T. Kinoshita, P. Cvitanović.

  T. Kinoshita, P. Cvitanović [1974]: $A_1^{(6)} = 1.195 \pm 0.026$
- E. Remiddi, S. Laporta et al., ~1965…1996, analytically: $A_1^{(6)} = 1.181241456\ldots$
- T. Kinoshita et al., numerically, 2015: $A_1^{(8)} = -1.91298(84)$
- T. Kinoshita et al., numerically, 2015: $A_1^{(10)} = 7.795(336)$
- S. Laporta, semi-analytically, 2017: $A_1^{(8)} = -1.9122457649\ldots$

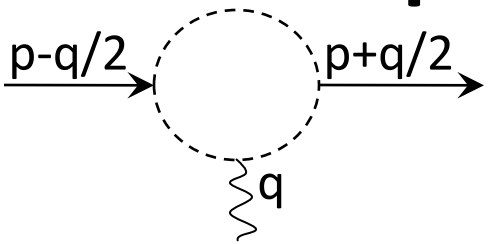
The method

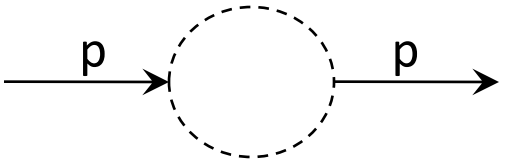
- Subtraction procedure for removing both IR and UV divergences in Feynman-parametric space for each individual Feynman diagram
- Diagram-specific importance sampling Monte Carlo integration algorithm for diagrams without lepton loops

The subtraction procedure

- FULLY AUTOMATED AT ANY ORDER OF THE PERTURBATION SERIES.
- UV and IR divergences are eliminated point-by-point in Feynman-parametric space for each individual Feynman diagram. No regularization is required.
- Subtraction by a forest formula with linear operators. Each operator transforms Feynman amplitude of some UV-divergent subdiagram G’ (in momentum space) to the polynom with the degree that is less or equal to $\omega(G')$.
- The subtraction is equivalent to the on-shell renormalization => no residual renormalizations, no calculations of renormalization constants, no other manipulations.

Operators

$\Gamma_\mu(p,q) =$ 

$\Sigma(p) =$ 

- A – projector of AMM

$$\bar{u}_2\Gamma_\mu(p,q)u_1 = \bar{u}_2(f(q^2)\gamma_\mu - g(q^2)\sigma_{\mu\nu}q^\nu/(2m) + h(q^2)q_\mu)u_1$$

$$\sigma_{\mu\nu} = (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/2, \qquad (p-q/2)^2 = (p+q/2)^2 = m^2$$

$$(\hat{p} - \hat{q}/2 - m)u_1 = (\hat{p} + \hat{q}/2 - m)u_2 = 0$$

$$A\Gamma_\mu = \gamma_\mu \lim_{q^2\to 0} g(q^2)$$

- U – intermediate operator

$$\Gamma_\mu(p,0) = a(p^2)\gamma_\mu + b(p^2)p_\mu + c(p^2)\hat{p}p_\mu + d(p^2)(\hat{p}\gamma_\mu - \gamma_\mu\hat{p}) \qquad \Sigma(p) = r(p^2) + s(p^2)\hat{p}$$

$$U\Gamma_\mu = \gamma_\mu a(m^2) \qquad U\Sigma = r(m^2) + s(m^2)\hat{p}$$

U preserves the Ward identity!

For the other types of divergent subgraphs, U=Taylor expansion at 0 up to ω order.

- L – on-shell renormalization for vertex-like subdiagrams

$$L\Gamma_\mu = \gamma_\mu(a(m^2) + b(m^2)m + c(m^2)m^2)$$

Forest formula for AMM

A set of subgraphs of a diagram is called a **forest** if any two elements of this set don't overlap.

$\mathcal{F}$[G] – the set of all forests of UV-divergent subgraphs in G that contain G.

$\mathbb{I}$[G] – the set of all vertex-like UV-divergent subgraphs in G that contains the vertex that is incident to the external photon line of G.

$$\tilde{f}_G = \sum_{\substack{F=\{G_1,\ldots,G_n\}\in\mathcal{F}[G] \\ G'\in\mathbb{I}[G]\cap F}} (-1)^{n-1} K_{G_1}^{G'} \ldots K_{G_n}^{G'} f_G$$

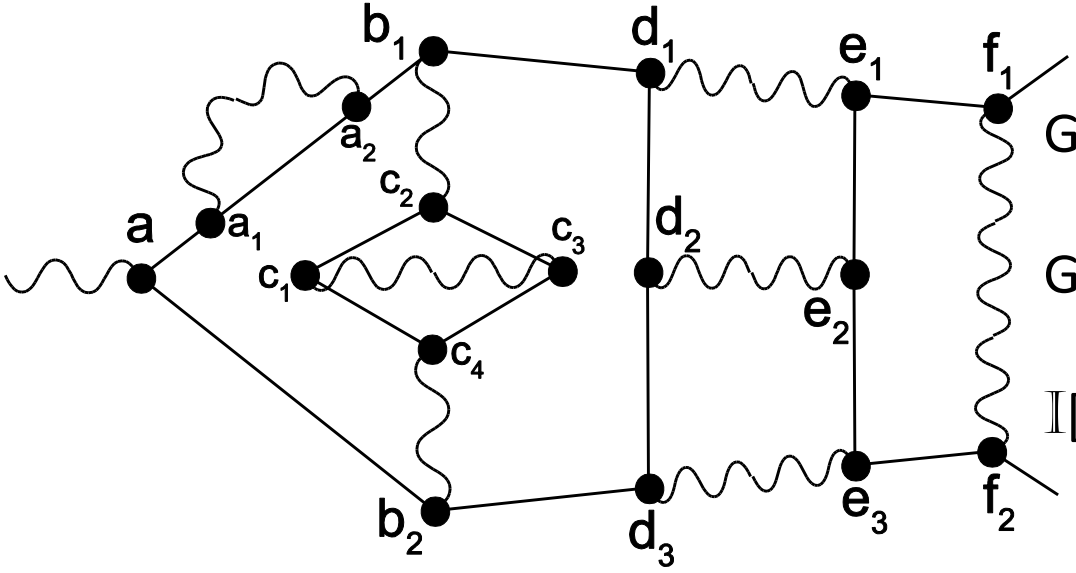
$$K_{G''}^{G'} = \begin{cases} A_{G'} \text{ for } G' = G'' \\ U_{G''} \text{ for } G'' \notin \mathbb{I}[G], \text{ or } G'' \subseteq G' \text{ and } G'' \neq G' \\ L_{G''} \text{ for } G'' \in \mathbb{I}[G],\ G' \subseteq G'',\ G'' \neq G,\ G'' \neq G' \\ (L_{G''} - U_{G''}) \text{ for } G'' = G,\ G' \neq G \end{cases}$$

$$\bar{f}_G = \text{coefficien t before } \gamma_\mu \text{ in } \tilde{f}_G$$

$$a_e = \sum_G \bar{f}_G$$

Details: S. Volkov, J. Exp. Theor. Phys. (2016), V. 122, N. 6, pp. 1008-1031

Example



$G_c=aa_1a_2b_1b_2c_1c_2c_3c_4$

$G_e=aa_1a_2b_1b_2c_1c_2c_3c_4d_1d_2d_3e_1e_2e_3$

$\mathbb{I}[G]=\{G_c,G_e,G\}$

Other UV-divergent subgraphs:

electron self-energy – a_1a_2, vertex-like – $c_1c_2c_3$, $c_1c_3c_4$,

photon self-energy – $c_1c_2c_3c_4$,

photon-photon scattering – $G_d=aa_1a_2b_1b_2c_1c_2c_3c_4d_1d_2d_3$

$$\tilde{f}_G=\left[A_G(1-U_{G_e})(1-U_{G_c})-(L_G-U_G)A_{G_e}(1-U_{G_c})-(L_G-U_G)(1-L_{G_e})A_{G_c}\right]\cdot$$
$$\cdot(1-U_{G_d})(1-U_{c_1c_2c_3c_4})(1-U_{c_1c_2c_3}-U_{c_1c_3c_4})(1-U_{a_1a_2})f_G$$

Importance sampling Monte Carlo

- Integral: $\int_{\Omega} f(x)dx$
- Probability density function: $g(x)$
- Approximation: $(1/N)\Sigma_{1\leq j\leq N}(f(x_j)/g(x_j))$
- Variance: $V(f,g)= \int_{\Omega}(f(x)^2/g(x))dx-(\int_{\Omega}f(x)dx)^2$
- Error estimation: $\sigma^2 \approx V(f,g)/N$
- The goal is to minimize $V(f,g)$ by choosing $g(x)$.

Diagram-specific probability density functions

- Integral: $\int_{z_1,\ldots,z_M>0} f(z_1,\ldots,z_M)\delta(z_1+\ldots+z_M-1)dz$
- Sector: $z_{j_1} \geq z_{j_2} \geq \ldots \geq z_{j_M}$
- Density: $C \cdot \frac{\prod_{l=2}^{M} (z_{j_l} / z_{j_{l-1}})^{Deg(\{j_l, j_{l+1},\ldots,j_M\})}}{z_1 \cdot z_2 \cdot \ldots \cdot z_M},$

Deg is defined on subsets of {1,…,M}

(the idea of E.Speer, J. Math. Phys. 9, 1404 (1968))

- My ideas are:

1) how to calculate *Deg*(s) for each set s
   (taking into account the infrared behavior etc.)
2) how to generate samples fastly

Obtaining *Deg*(s)

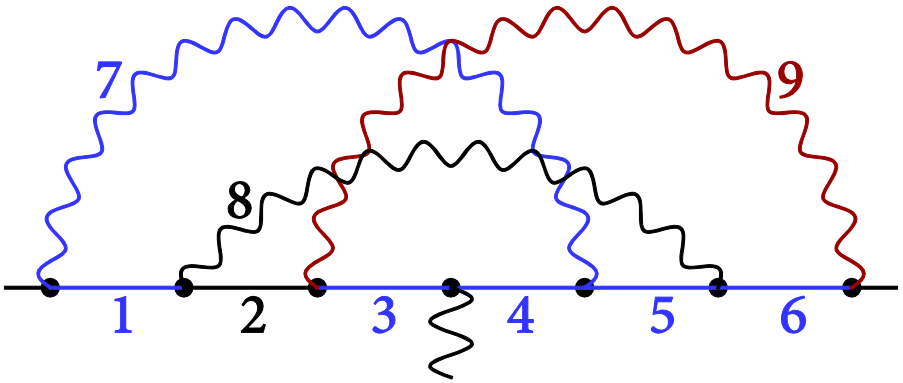
- Sector: $z_{j_1} \geq z_{j_2} \geq \ldots \geq z_{j_M}$
- Density: $C \cdot \dfrac{\prod_{l=2}^{M} (z_{j_l} / z_{j_{l-1}})^{Deg(\{j_l, j_{l+1}, \ldots, j_M\})}}{z_1 \cdot z_2 \cdot \ldots \cdot z_M},$

•The rules are constructed using ultraviolet degrees of divergence (with the sign '-') of I-closures of sets

(the full description taking into account divergent subdiagrams is in https://arxiv.org/abs/1705.05800)

•I$\mathrm{Clos}(s)$=sUs', where *s'* is the set of all photon lines for which the electron path connecting their ends is contained in *s*

Example: IClos({1,3,4,5,6,7})={1,3,4,5,6,7,9}

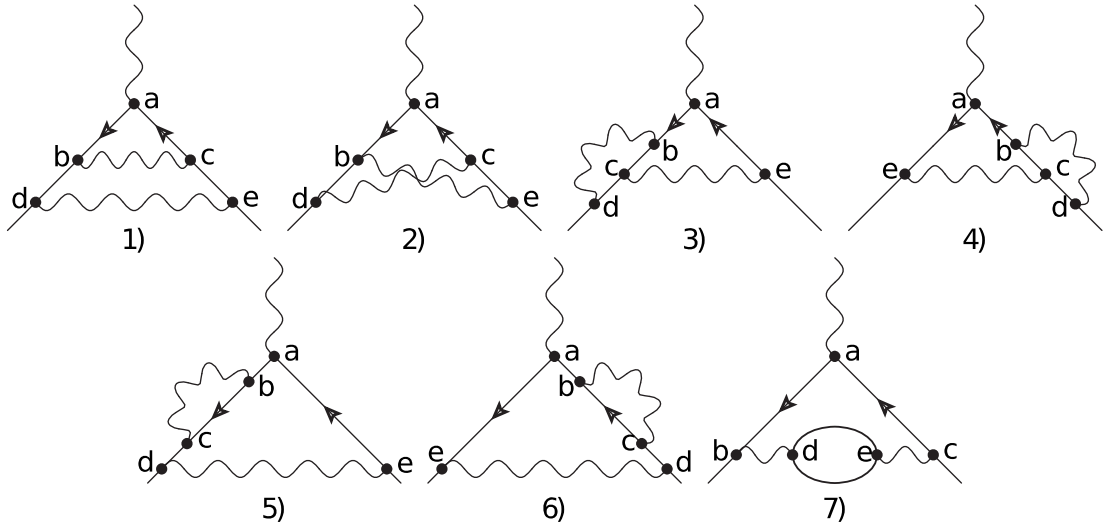


Numerical results (for testing the method)

- 2 loops: all Feynman diagrams (old, 2015)
- 3 loops: all Feynman diagrams (old, 2015)
- 4 loops: diagrams without electron loops (new)
- up to 5 loops: ladder diagrams (new)
- up to 5 loops: cross diagrams (new)
- comparison with the other calculations with respect to the Monte Carlo convergence speed

2 loops: all Feynman diagrams (2015)

without new Monte-Carlo

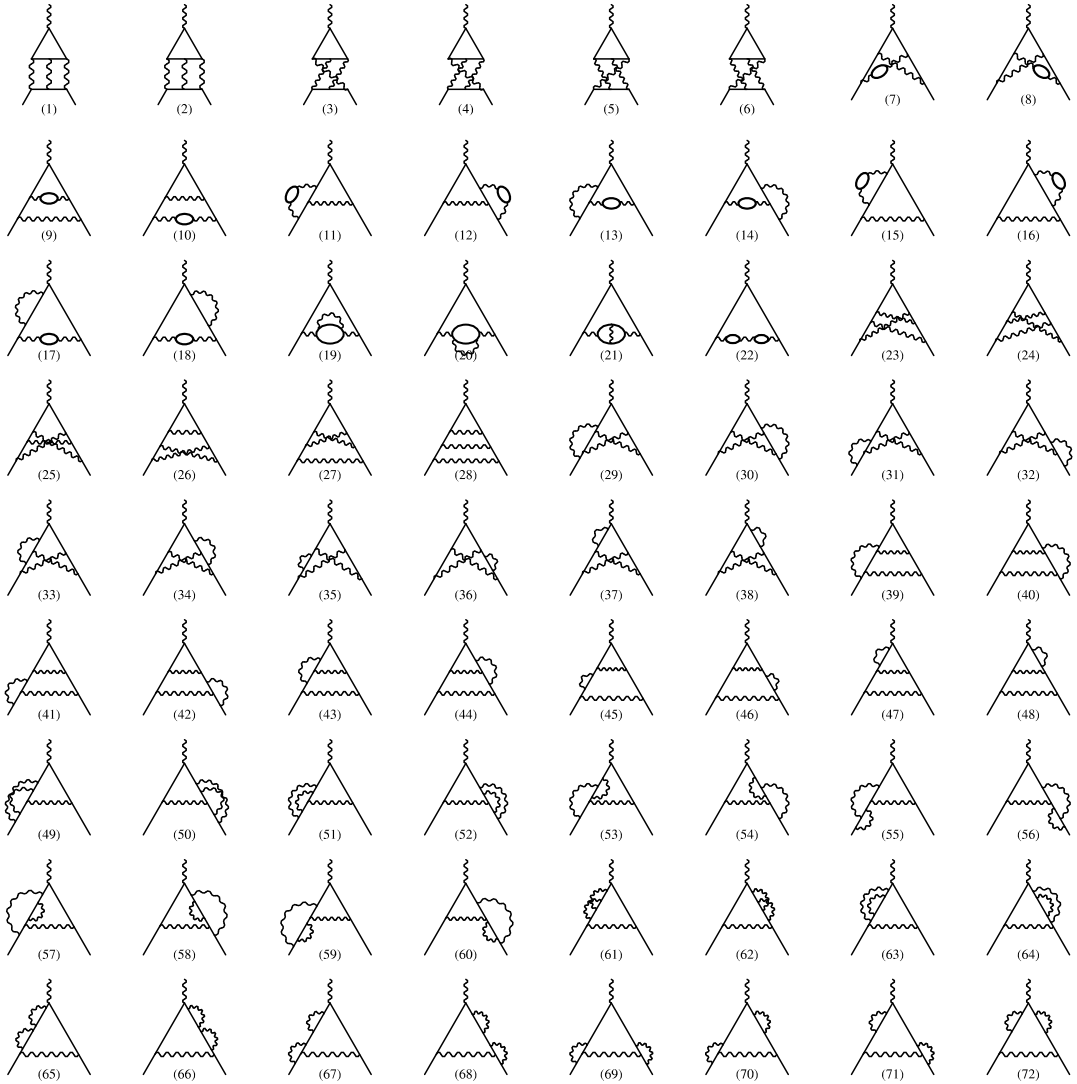


#	My value	Analytical value (Petermann, 1957)
1	0.777455(52)	0.777478
2	-0.467626(44)	-0.467645
3	-0.032023(29)	$-0.282-(1/4)\log(\lambda^2/m^2)$
4	-0.032033(29)	$-0.282-(1/4)\log(\lambda^2/m^2)$
5	-0.294978(25)	$-0.045+(1/4)\log(\lambda^2/m^2)$
6	-0.294998(24)	$-0.045+(1/4)\log(\lambda^2/m^2)$
7	0.0156895(25)	0.0156874

My result: $A_1^{(4)} = -0.328513(87)$

Analytical, 1957: $A_1^{(4)} = -0.328478966...$

3 loops: all Feynman diagrams (2015, without new Monte-Carlo)



3-loop Feynman diagrams for electron's AMM. Plot courtesy of F.Jegerlehner

Comparison with known analytical values

#	My value	Analyt. val.	Reference
1-6	0.3708(14)	0.3710	[10]
7-10	0.04989(20)	0.05015	[4,5]
11-12,15-16	-0.08782(15)	-0.08798	[2,4]
13-14,17-18	-0.11230(17)	-0.11234	[3,4]
19-21	0.05288(13)	0.05287	[1]
22	0.002548(20)	0.002559	[1]
23-24	1.8629(14)	1.8619	[11]
25	-0.02688(47)	-0.02680	[12]
26-27	-3.1764(22)	-3.1767	[8]
28	1.7888(19)	1.7903	[8]
29-30	-1.7579(10)	-1.7579	[12]
31-32,37-38	5.3559(27)	5.3576	[8,9]
33-34,37-38	0.4549(14)	0.4555	[8,11]
31-32,35-36	1.5436(34)	1.5416	[7,9]
33-36	-3.3573(24)	-3.3605	[7,11]
39-40	-0.33468(95)	-0.33470	[11]
41-48	-0.4030(41)	-0.4029	[6,7]
49-72	0.9529(53)	0.9541	[6-9,11,12]

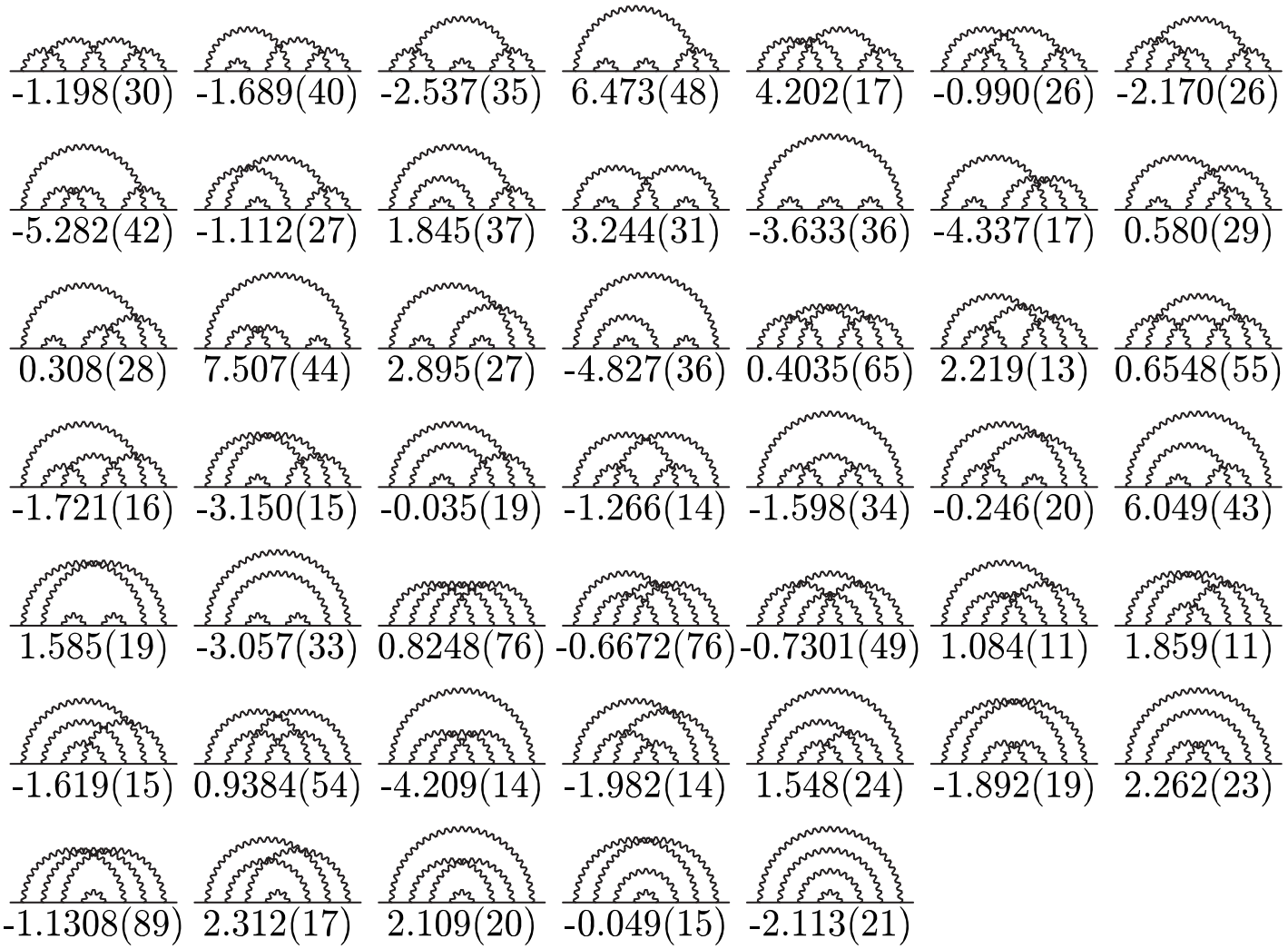
[1] J. Mignaco, E. Remiddi, IL Nuovo Cimento, V. LX A, N. 4, 519 (1969).
[2] R. Barbieri, M. Caffo, E. Remiddi, Lettere al Nuovo Cimento, V. 5, N. 11, 769 (1972).
[3] D. Billi, M. Caffo, E. Remiddi, Lettere al Nuovo Cimento, V. 4, N. 14, 657 (1972).
[4] R. Barbieri, E. Remiddi, Physics Letters, V. 49B, N. 5, 468 (1974).
[5] R. Barbieri, M. Caffo, E. Remiddi, Ref.TH.1802-CERN (1974).
[6] M. Levine, R. Roskies, Phys. Rev. D, V. 9, N. 2, 421 (1974).
[7] M. Levine, R. Perisho, R. Roskies, Phys. Rev. D, V. 13, N. 4, 997 (1976).
[8] R. Barbieri, M. Caffo, E. Remiddi et al., Nuclear Physics B 144, 329 (1978).
[9] M. Levine, E. Remiddi, R. Roskies, Phys. Rev. D, V. 20, N. 8, 2068 (1979).
[10] S. Laporta, E. Remiddi, Physics Letters B 265, 182 (1991).
[11] S. Laporta, Physics Letters B 343, 421 (1995).
[12] S. Laporta, E. Remiddi, Physics Letters B 379, 283 (1996).

4 loops: diagrams without electron loops (new)

My result: -2.34(17) 2 weeks, 2 CPU cores

Kinoshita et al., 2012: -2.17550(194)

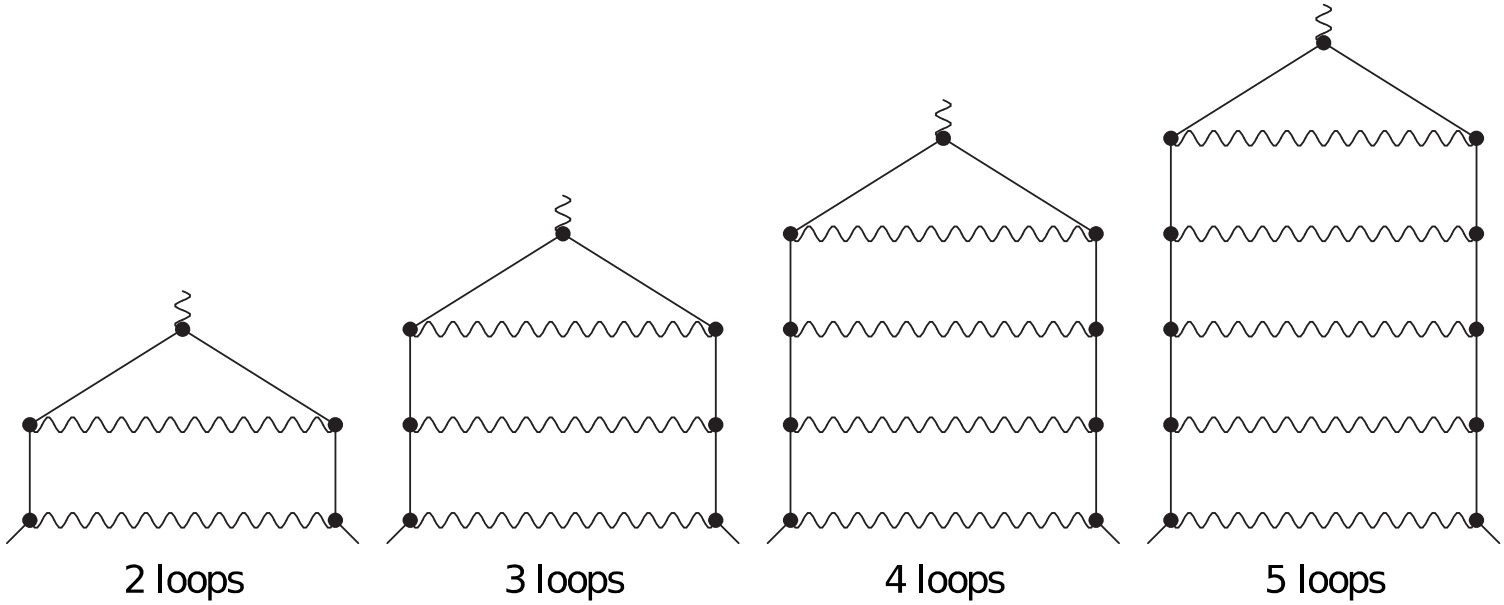
Laporta, 2017: -2.1768660277…



Ladder diagrams: up to 5 loops

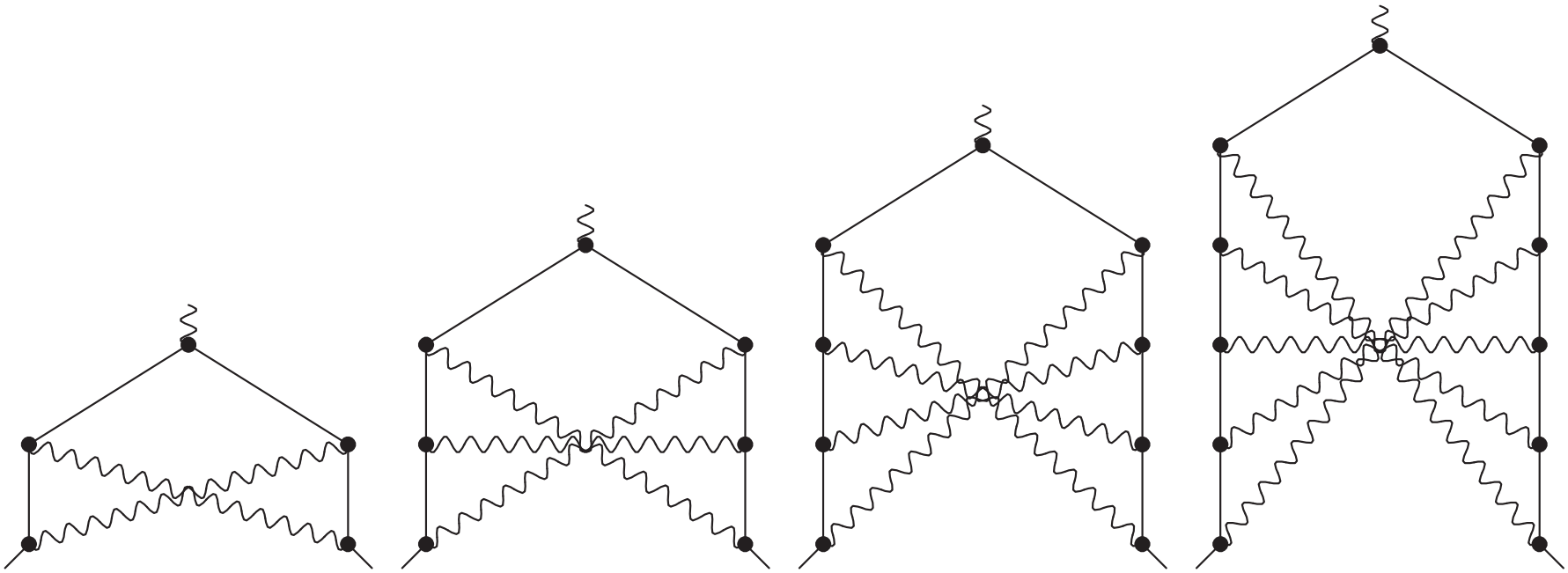
loops	My value	Analytical value	N_{call}	time
2	0.777440(67)	0.777478…	$62 \cdot 10^7$	3 hours
3	1.79052(37)	1.790278…	$68 \cdot 10^7$	11 hours
4	4.3035(39)	4.29765…	$39 \cdot 10^7$	24 hours
5	11.531(78)	11.6592…	10^8	45 hours

All analytical values are from M. Caffo, S. Turrini, E.Remiddi, Nuclear Physics B141 (1978) 302-310.



Cross diagrams: up to 5 loops

loops	My value	Analytical value	N_{call}	time
2	-0.467666(49)	-0.467645…	$61 \cdot 10^7$	2.5 hours
3	-0.026810(47)	-0.026800…	$56 \cdot 10^7$	10 hours
4	0.29685(21)	-	$14 \cdot 10^7$	26 hours
5	-0.6421(34)	-	10^7	24 hours



2 loops 3 loops 4 loops 5 loops

Comparison with the other methods with respect to Monte Carlo convergence speed

Not quite correct, but…

Calculation	σ	N_{call}	$\sigma \cdot \sqrt{N_{\text{call}}}$
3 loops, my method, 8 integrands	0.0055	$43 \cdot 10^7$	119.2
3 loops, Carroll&Yao 1974 [1], 8 integrands, RIWIAD	0.06	$16 \cdot 10^6$	240
3 loops, Kinoshita et al. 1995 [2], 8 integrands, VEGAS	0.000347	$3 \cdot 10^{12}$	601
4 loops, my method, 47 integrands	0.17	10^9	5375.9
4 loops, Kinoshita et al. 2006 [3], 47 integrands, VEGAS	0.00343	$87 \cdot 10^{12}$	31992.9

[1] R. Carroll, Y. Yao, Physics Letters, 1974, V. 48B, N. 2, pp. 125-127.
[2] T. Kinoshita, Physical Review Letters, 1995, V. 75, N. 21, pp. 4728-4731.
[3] T. Kinoshita, M. Nio, Physical Review D 73, 013003 (2006).

Thank you
for your attention!

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https://arxiv.org/abs/1705.05800