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$\overline{\mathrm{MS}}$ and on-shell QED $O(\alpha^6)$ expression for $\sigma(e^+e^- \to \gamma \to l_i^+ l_i^-)$ and consequences of the conformal symmetry limit

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Based on INR-TH-2017-007, by Garkusha, Kataev and Molokoedov, work in progress

$\sigma(e^+e^- \to \gamma^* \to l_i^+ l_i^-)$ is inspired by

Introduction

1. Up to $O(\alpha^4)$ results may be of interest to experimentalists : τ-lepton factory in Novosibirsk , Private information by **Roman Lee**

2. Theoretical argument for applying $\overline{\mathrm{MS}}$-scheme in phenomenological QED studies: $\alpha_{\overline{\mathrm{MS}}}(m_{l_i}) = \alpha_{OS}(m_{l_i})\big(1 + O(\alpha^2_{OS})\big)$

3. Analytical structure of the QED version the QCD corrections to $O(\alpha_s^4)$ $SU(N_c)$ expressions for $R^{e^+e^-}$-ratio
α_s^2- analytical- **Chetyrkin,Kataev,Tkachov(79)**; numerical- **Dine,Sapirstein(79)** ; α_s^3- analytical **Gorishny,Kataev,Larin(91);Surguladze,Samuel(91)**; α_s^4-analytical **Baikov,Chetykin,Kuhn(10)+Rittinger(12)**;
Peculiar asymptotic structure of studied QED PT series in $\overline{\mathrm{MS}}$ and OS schemes

4. Identification of the scheme-independent contributions, surviving in respecting conformal symmetry "finite QED program" **Johnson,Baker,Willey(67),Adler,Callan,Gross,Jakiw(72)**, **Kataev(08,14)** In QCD Principle of Maximal Conformality results for R-ratio **Brodsky,Wu et al(14)-(17)**, differ from **Kataev,Mikhailov(15,16)**. QED results to be presented can be used to support results of **Kataev,Mikhailov(15,16), Cvetic,Kataev(16)**

Definitions and renormalisation group equations

The studied QED physical quantities are defined as

$$\sigma(e^+e^- \to \gamma \to l_i^+ l_i^-) = \frac{4\pi^2 a}{3s} F_1(v_{l_i}) R^{e^+e^-}(s)|_{SU(N_c)\to U(1)}$$

Here $a = \alpha/\pi$, $F_1(v_{l_i}) = v_{l_i}(3 - v_{l_i}^2)/2$, $v_{l_i} = \sqrt{1 - 4m_{l_i}^2/s}$, m_{l_i} with $i = 1, 2, 3$ are the pole masses of electron, muon and τ-lepton and $R^{e^+e^-}(s)|_{SU(N_c)\to U(1)}$ is defined in QED as

$$R^{e^+e^-}(s)|_{QED} = 1 + \sum_{i=1}^{4} \left(r_i + O(\frac{m_{l_i}^2}{s}) \right) a(s)^i + O(a^5)$$

r_i are obtained from QCD analytical expressions of in the $\overline{\rm MS}$-scheme fixing $\rm C_A = 0$, $\rm C_F = 1$, $\rm T = 1$, $n_f = N_{l_i}$, $d^{abc}d^{abc}/d_R = 1$, $d^{abcd}d^{abcd}/d_R = 1$ and $a(s) = a_{\overline{\rm MS}}(s) = \frac{\alpha_{\overline{\rm MS}}(s)}{\pi}$.

In the massless limit this quantity obeys the following RG equation

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(a) \frac{\partial}{\partial a} \right) \sigma(e^+e^- \to \gamma \to l_i^+ l_i^-)(a(\mu^2, s/\mu^2)) = 0$$

where QED β-function is known at present at the 5-loop level $\beta(a) = \mu^2 \frac{\partial a}{\partial \mu^2} = \sum_{i=0}^{4} \beta_i a^{i+2}$ in the $\overline{\rm MS}$ and OS-scheme (**Baikov,Chetykin,Kuhn,Rittinger(12)** for arbitrary $N = N_{l_i}$ **Kataev,Larin(12)** for $N = 1$)

The analytical and numerical expressions for r_i will be presented in the $\overline{\rm MS}$ and *OS*-schemes

$\overline{\mathrm{MS}}$ and *OS* results and CS scheme-independent (SI) consequences

$$\sigma(e^+e^- \to \gamma \to l_i^+ l_i^-) = \frac{4\pi^2 a}{3s} F_1(v_{l_i})\left[1 + \sum_{i=1}^4 \left(r_i + O(\frac{m_{l_i}^2}{s})\right) a(s)^i\right]$$ **red**

= respecting CS ; SI renormalon + SI analytic continuation ;

green= not yet understood coincidence

$$r_1^{\overline{\mathrm{MS}}} = \frac{3}{4} \quad , \mathbf{r}_2^{\overline{\mathrm{MS}}} = -\frac{3}{32} + \left(-\frac{11}{8} + \zeta_3\right)\mathbf{N}$$

$$r_1^{OS} = \frac{3}{4} \quad , \quad \mathbf{r}_2^{\mathbf{OS}} = -\frac{3}{32} + \left(-\frac{11}{8} + \zeta_3\right)\mathbf{N}$$

$$\begin{aligned} r_3^{\overline{\mathrm{MS}}} &= -\frac{69}{128} + \left(-\frac{29}{64} + \frac{19}{4}\zeta_3 - 5\zeta_5\right)\mathbf{N} \\ &+ \left(\frac{11}{192} - \frac{1}{8}\zeta_3\right)^{\mathbf{SI}}\mathbf{N} + \left(\frac{151}{54} - \frac{19}{9}\zeta_3\right)\mathbf{N}^2 - \frac{\pi^2}{12}\mathbf{N}^2 \\ &= -\frac{69}{128} - \left(\frac{19}{48} - \frac{37}{8}\zeta_3 + 5\zeta_5\right)\mathbf{N} + \left(\frac{151}{54} - \frac{19}{9}\zeta_3 - \frac{\pi^2}{12}\right)\mathbf{N}^2 \end{aligned}$$

$$\begin{aligned} r_3^{OS} &= -\frac{69}{128} + \left(\frac{1}{4} + \frac{19}{4}\zeta_3 - 5\zeta_5\right)\mathbf{N} \\ &+ \left(\frac{11}{192} - \frac{1}{8}\zeta_3\right)^{\mathbf{SI}}\mathbf{N} + \left(\frac{151}{54} - \frac{19}{9}\zeta_3\right)\mathbf{N}^2 - \frac{\pi^2}{12}\mathbf{N}^2 \\ &= -\frac{69}{128} + \left(\frac{59}{192} + \frac{37}{8}\zeta_3 - 5\zeta_5\right)\mathbf{N} + \left(\frac{151}{54} - \frac{19}{9}\zeta_3 - \frac{\pi^2}{12}\right)\mathbf{N} \end{aligned}$$

Continuation- $(\frac{\alpha}{\pi})^4$ coefficient in $\overline{\mathrm{MS}}$ and OS-schemes: structure

$$
\begin{aligned}
r_4^{\overline{\mathrm{MS}}} &= \left(\frac{4157}{2048}+\frac{3}{8}\zeta_3\right)+\left(\frac{1001}{384}+\frac{99}{32}\zeta_3-\frac{125}{4}\zeta_5+\frac{105}{4}\zeta_7\right)\mathrm{N} \\
&+ \left(-\frac{13}{16}-\zeta_3+\frac{5}{2}\zeta_5\right)^{\mathrm{lbl}}\mathrm{N}+\left(-\frac{13}{64}-\frac{1}{4}\zeta_3+\frac{5}{8}\zeta_5\right)^{\mathrm{SI}}\mathrm{N} \\
&+ \left(\frac{5713}{1728}-\frac{581}{24}\zeta_3+\frac{125}{6}\zeta_5+3\zeta_3^2\right)\mathrm{N}^2+\underline{\left(-\frac{149}{576}+\frac{13}{32}\zeta_3-\frac{5}{16}\zeta_5+\frac{1}{8}\zeta_3^2\right)^{\mathrm{SI}}\mathrm{N}^2} \\
&- \underline{\frac{7\pi^2}{288}\mathrm{N}^2+\frac{\pi^2}{9}\left(\frac{11}{8}-\zeta_3\right)\mathrm{N}^3-\left(\frac{6131}{972}-\frac{203}{54}\zeta_3-\frac{5}{3}\zeta_5\right)\mathrm{N}^3} \\
r_4^{OS} &= \left(\frac{4157}{2048}+\frac{3}{8}\zeta_3\right)+\left(\frac{81}{32}+\frac{793}{256}\zeta_3-\frac{125}{4}\zeta_5+\frac{105}{4}\zeta_7+\frac{15}{16}\zeta_2-\frac{3}{2}\zeta_2\ln 2\right)\mathrm{N} \\
&+ \left(-\frac{13}{16}-\zeta_3+\frac{5}{2}\zeta_5\right)^{\mathrm{lbl}}\mathrm{N}+\left(-\frac{13}{64}-\frac{1}{4}\zeta_3+\frac{5}{8}\zeta_5\right)^{\mathrm{SI}}\mathrm{N}+\left(-\frac{11}{144}-\frac{17089}{768}\zeta_3\right. \\
&+ \left.\frac{125}{6}\zeta_5+3\zeta_3^2+\frac{1}{6}\zeta_2\right)\mathrm{N}^2+\underline{\left(-\frac{149}{576}+\frac{13}{32}\zeta_3-\frac{5}{16}\zeta_5+\frac{1}{8}\zeta_3^2\right)^{\mathrm{SI}}\mathrm{N}^2} \\
&- \underline{\frac{7\pi^2}{288}\mathrm{N}^2+\frac{\pi^2}{9}\left(\frac{11}{8}-\zeta_3\right)\mathrm{N}^3-\left(\frac{6131}{972}-\frac{203}{54}\zeta_3-\frac{5}{3}\zeta_5\right)\mathrm{N}^3}
\end{aligned}
$$

Continuation: α^4 coefficient in $\overline{\mathrm{MS}}$ and *OS*-schemes: results

$$
\begin{aligned}
r_4^{\overline{\mathrm{MS}}} &= \frac{4157}{2048} + \frac{3}{8}\zeta_3 + \left(\frac{611}{384} + \frac{59}{32}\zeta_3 - \frac{225}{8}\zeta_5 + \frac{105}{4}\zeta_7\right)N + \left(\frac{2633}{864} - \frac{2285}{96}\zeta_3 + \frac{985}{45}\zeta_5\right. \\
&+ \left.\frac{25}{8}\zeta_3^2 - \frac{7\pi^2}{288}\right)N^2 - \left(\frac{6131}{972} - \frac{209}{54}\zeta_3 - \frac{5}{3}\zeta_5 - \frac{11\pi^2}{72} + \frac{\pi^2\zeta_3}{9}\right)N^3 \\
r_4^{OS} &= \frac{4157}{2048} + \frac{3}{8}\zeta_3 + \left(\frac{97}{64} + \frac{473}{256}\zeta_3 - \frac{225}{8}\zeta_5 + \frac{105}{4}\zeta_7 + \frac{15}{16}\zeta_2 - \frac{3}{2}\zeta_2\ln(2)\right)\mathrm{N} \\
&+ \left(-\frac{193}{575} - \frac{16777}{768}\zeta_3 + \frac{985}{48}\zeta_5 + \frac{25}{8}\zeta_3^2 + \frac{\zeta_2}{2} - \frac{7\pi^2}{288}\right)N^2 \\
&+ \left(\frac{6131}{972} + \frac{203}{54}\zeta_3 + \frac{5}{3}\zeta_5 + \frac{11\pi^2}{72} - \frac{\pi^2\zeta_3}{9}\right)N^3
\end{aligned}
$$

The expressions are more complicated- difficult to analyse without numerical representations

Numerical results for $\overline{\mathrm{MS}}$-scheme and *OS* scheme

$$
\begin{aligned}
r(s)_e^{\overline{\mathrm{MS}}} &= 1 + 0.75 a_{\overline{\mathrm{MS}}}(s) - 0.2667 a_{\overline{\mathrm{MS}}}(s)^2 - 1.124_{\overline{\mathrm{MS}}} a(s)^3 + 4.5253 a_{\overline{\mathrm{MS}}}(s)^4 \\
r(s)_e^{OS} &= 1 + 0.75 a_{OS}(s) - 0.2667 a_{OS}(s)^2 - 0.4207_{OS}\, a(s)^3 + 3.2663 a_{OS}(s)^4
\end{aligned}
$$

$$
\begin{aligned}
r(s)_\mu^{\overline{\mathrm{MS}}} &= 1 + 0.75 a_{\overline{\mathrm{MS}}}(s) - 0.3496 a_{\overline{\mathrm{MS}}}(s)^2 - 2.826 a_{\overline{\mathrm{MS}}}(s)^3 + 8.9899 a_{\overline{\mathrm{MS}}}(s)^4 \\
r(s)_\mu^{OS} &= 1 + 0.75 a_{OS}(s) - 0.3496 a_{OS}(s)^2 - 1.14301 a_{OS}(s)^3 + 4.39172 a_{OS}(s)^4
\end{aligned}
$$

$$
\begin{aligned}
r(s)_\tau^{\overline{\mathrm{MS}}} &= 1 + 0.075 a_{\overline{\mathrm{MS}}}(s) - 0.6126 a^2_{\overline{\mathrm{MS}}}(s) - 5.6765 a^3_{\overline{\mathrm{MS}}}(s) + 16.619 a_{\overline{\mathrm{MS}}}(s)^4 \\
r(s)_\tau^{OS} &= 1 + 0.075 a_{OS}(s) - 0.6126 a_{OS}(s)^2 - 3.5672 a_{OS}(s)^3 + 6.6315 a_{OS}(s)^4
\end{aligned}
$$

Conclusion: in the *OS* scheme starting from $O(a_s^3)$ in QED PT series for $r(s)$ behaves better, then in the $\overline{\mathrm{MS}}$-scheme; at the the $O(a^2)$ level coefficients coincide

General Conclusion

1. The analytical and numerical $O(a^5)$ QED results for the model physical quantity $\sigma(e^+e^- \to \gamma \to l_i^+ l_i^-)$ in the $\overline{\mathrm{MS}}$ and OS scheme are presented, they should be useful in practical studies after calculating additional diagrams, at the $O(a^4)$- level, box-type with masses

2. The arguments in favour of advantage of application of the OS physical scheme prior $\overline{\mathrm{MS}}$-scheme are presented, though at the NLO these schemes give identical result and at the NNLO differ by one rational contribution only, but this difference is physically important

3. The general structure of analytical $O(a^6)$-results are analysed and the scheme-independent and respecting conformal symmetry analytical contributions are separated. Interesting from theoretical point of view.

4. It is demonstrated that at the 5-loop level QED expressions for $r(s)$, equivalent to $O(a^6)$ approximation of $\sigma(e^+e^- \to \gamma \to l_i^+ l_i^-)$, perturbation theory series are not sign alternating (different from $(g-2)_e$ and theoretical expectations of the behaviour of asymptotic structure of PT series in QED)